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The quasi-similarity and dominant conditions of Weyl spectrum in a complex Hilbert space

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Abstract
The study of spectrum in Hilbert spaces is a very rich in giving more structures of the spectrum and we wish to go in to depth to understand deeper on the structure of the spectrum. Apart from the well-known components of spectrum i.e. spectrum, the approximate point spectrum, the point spectrum and the set of eigenvalues of finite multiplicity: there is need for further study on the Weyl spectrum of an Operator in a complex Hilbert space. To succeed in this study, two conditions to help expose properties of the Weyl spectrum i.e. Quasi-Similarity and the dominant condition will be used.

Key words: Dominant, fredholm, invertible, isometry, quasi-similarity.

INTRODUCTION

Let H be a complex Hilbert space. By an operator on H, we shall mean a bounded linear transformation from H to H. Let $\sigma(\mathrm{T})$, $\pi(\mathrm{T})$, $\pi_0(\mathrm{T})$. $\pi_{00}(\mathrm{T})$ and $\omega(\mathrm{T})$, respectively denote the spectrum, the approximate point spectrum, the point spectrum, the set eigenvalues with finite multiplicity and the Weyl spectrum of an operator T. If for an operator T, $\omega(\mathrm{T}) = \sigma(\mathrm{T}) \sim \pi_{00}(\mathrm{T})$, then we say that the Weyl's theorem holds for T. The spectral radius and the numerical radius of T will be denoted by $\mathrm{r}(T)$. If $\mathrm{r}(T) = |\mathrm{W}(T)|$, then T is exists some $M > 0$ such that $\|T^{n}\| \leq M$ for each positive integer n. According to Berberian (1969), an operator is defined to be a 2 isometry if $T^{*2}T^2 - 2T^*T + I = 0$. In the present note, we explore some properties of 2-isometries. Clearly every isometry is a 2-isometry. Now an invertible 2-isometry turns out to be a unitary operator. It is obvious from the definition that every 2-isometry is invertible. In particular if both $T$and T^* are 2-isometries then T is invertible and so must be unitary.

RESULTS AND DISSCUSSIONS

Isometries and Dominance on Operators

It's important to discuss on isometry and dominance to help us determine where the unilateral weighed shifts lies in the unit circle. We start by stating the following theorem and proofing it.

Theorem 1

A power of a 2-isometry is again a 2-isometry.

PROOF. Let T be a 2-isometry. We prove the assertion by using the mathematical induction. Since T is a 2-2-isometry, the result is true for $\mathrm{n} = 1$. Now assume that the result is true for $\mathrm{n} = \mathrm{k}$,i.e.

(2.1)

$$T^{*2k}T^{2k} - 2T^{*k}T^{k} + 1 = 0.$$

Then

$$T^{*2(k+1)}T^{2(k+1)} - 2T^{*k+1}T^{k+1} + I$$
$$= T^{*2}(T^{2k}T^{2k})T^2 - 2T^{*k+1}T^{k+1} + I$$
$$= T^{*2}(2T^{*k}T^{k} - 1)T^2 - 2T^{*k+1}T^{k+1} + I$$
$$\text{(by (2.1))}$$

$$= 2T^{*K+2}T^{k+2} - T^{*2}T^2 - 2T^{*K+1}T^{K+1} + I$$

$$= 2T^{*K}(T^*T - I)T^{K} - T^{*2}T^2 + I$$ (T is a 2-isometry)
$$= (T^{*k+1}T^{K+1} - 2T^{*K}T^{K}) - T^{*2}T^2 + I$$
$$= 2(T^{*2}T^2 - T^*T) - T^{*2}T^2 + I \; (by\ (2.1))$$
$$= T^*T^2 - 2T^*T + I$$
$$= 0.$$

This shows that the result is true for $\mathrm{n} = \mathrm{k} + 1$: thus T^n is a 2-isometry for each n. □

It is well known and obvious that a unilateral weighed shift is an isometry if all its weights lies on the unit circle. In the next result, we obtain a necessary and sufficient condition under which a non-isometric unilateral weighted shift is a 2-isometry.

Theorem 2

A non-isometric unilateral weighted shift $T$with weights $\{\propto_0\}$ is a 2-isometry if and only if

(i) $|\propto_n|^2|\propto_{n+1}|^2 - 2|\propto_n|^2 + 1 = 0$ for each n;

(ii) $|\propto_n| \neq 1$ for each n;

Proof

Suppose T is a 2-isometry. If $\{e_n\}$ is an orthonormal base for H, then $Te_n = \propto_n e_{n+1}$ and hence (i) follows. Suppose (ii) is false. Select the least positive integer k such that $|\propto_k| = 1$. If $k > 1$, then (i) gives$|\propto_{k-1} = 1$ which is contrary to the selection of k. Therefore $|\propto_1 = 1$. Using the induction argument and (i), one can show that $|\propto_n = 1$ for each positive integer n.

But this will contradict our assumption that T is non-isometric. Hence we conclude that (ii) is true. The conserve assertion is obvious.

□

Corollary 1

Let T be a non-isometric unilateral weighted shift with weights $\{\propto_n\}$. If T is a *2-isometry, then the following assertions hold.*

i. $\{|\propto_n|\}$ *is a strictly decreasing sequence of real numbers converging to 1.*

ii. $\sqrt{2} > |\propto_n| > 1$ *for each* $n > 1$.

Proof

(i) Suppose $|\propto_{n+1}| \geq |\propto_n|$ for some n. They be Theorem 2 (i), we find $0 \geq (1 - |\propto_n|^2)^2$ or $|\propto_n| = 1$. But this contradicts the numbers and so must be convergent.

Theorem 3

The spectrum of a 2-isometry is the closed unit disc provided it is non-unitary.

Proof

Let T be a non-unitary 2-isometry. Then $0 \in \sigma(T) \sim \pi(T)$. Since $\partial\sigma(T) \subseteq \pi(T), 0$ turns out to be an interior point of $\sigma(T)$. Therefore we can find the largest positive numbers r such that $\{2: |z| \leq r\}$ is contained in $\sigma(T)$ it is possible to select a complex number z in $\partial\sigma(T)$ such that $r = |z|$. Since $\partial\sigma(T) \subseteq \pi(T) \subseteq \{z: |z| = 1\}$ [1], $r = 1$. Consequently we find $\sigma(T) = \{z: |z| \leq 1\}$.

Corollary 2

If T is a 2-isometry, then each isolated point in its spectrum is an eigen-value.

Proof

If $\sigma(T)$ has an isolated point, then it is clear from the above theorem that T is unitary and hence the result follows.

□

Corollary 3

Let T be a 2-isometry. If the Lebesgue planar measure of $\sigma(T)$ is Zero, then T is unitary.

Corollary 4

The Weyl's theorem holds for 2-isometries.

Proof

The result holds if T is unitary. Assume that T is non-unitary. Then Theorem 1 shows that $\pi_{00}(T) = \phi$. Also by Theorem 2.9 (ii) and Lemma 3 of [2], $\sigma(T) \sim \pi_{00}(T) \subseteq \omega(T)$ and hence $\sigma(T) \subseteq \omega(T)$. This completes the argument

(i) we infer that $|\propto_n| \to 1$.

(ii) Rewriting equality (i) of theorem 2. as

It was known that $\omega(T)$ satisfies the one-way spectral mapping theorem for analytic functions: if f is analytic on neighborhood of $\sigma(T)$ then

(iii) $\omega(f(T) \subset f(\omega(T))$. (1.4)

(iv) If T is normal then $\sigma_e(T)$ and $\omega(T)$ coincide. Thus if T is normal since $f(T)$ is also normal, it follows that $\omega(T)$ satisfies the spectral mapping theorem for analytic functions. We say that Wely's theorem holds for (T)if

(v) $\omega(T) = \sigma(T) - \pi_{00}(T)$.

vi) It is well known that wely's theorem holds for any hyponormal operators, indeed, for any semi normal operator and for any Toeplitz operators (Berberian, 1969). Oberai (1974) has raised the following question: Does there exist a hyponormal operator T such that wely's theorem does not hold for T^2? Note that T^2 may not hyponormal even if T is hyponormal.

(vii) In this paper we show that the Weyl spectrum of a dominant operator satisfies the spectral mapping theorem for analytic functions, and that Weyl's theorem hold for $p(T)$ when T hyponormal and p is any polynormial. The latter result answers the question of Oberai.

Theorem 4

If S and $T$are dominant operators, then

$$S, T \text{ Weyl} \Leftrightarrow ST \text{ Weyl}.$$

Proof

If S, T are weyl, then S, T are Fredholm and $i(S) = i(T) = 0$. By Conway [3], ST is Fredholm and by the index product theorem, $i(ST) = i(S) + i(T) = 0$. Hence ST is Weyl.

Conversely if ST is Weyl, then ST is Fredholm and $i(ST) = 0$. Since S and T are dominant, $\ker S \subset \ker S^*$ and $\ker T \subset \ker T^*$ since $\ker S^* \subseteq \ker(ST)^*$, $\dim \ker S \leq \dim \ker S^* \leq \dim \ker (ST)^* < \infty$. Thus $\ker S$ and $\ker S^*$ are finite dimensional. By Schechter [10, Chap. 5 Theorem 3.5] S and T are Fredholm. Since $0 = i(ST) + i(S) + i(T)$ by the index product theorem, by (1.2) $i(S) = i(T) = 0$. Hence S and T are Weyl.

If the "dominant" condition is dropped in the above theorem, then the backward implication may fail even though T_1 and T_2 commute: For example, if U is the unilateral shift on l_2 consider the following operators on $l_2 \oplus l_2$: $T_1 = U \oplus I$ and $T_2 = I \oplus U^*$.

Proof

Suppose that p is any polynomial. Let

$$P(T) - \lambda I = a_o(T - \mu_1 I) \dots (T - \mu_n I).$$

Since T is dominant, $T - \mu_1 I$ are dominant operators for each $i = 1, 2, \dots, n$. It thus follows from Theorem 1 that

$$\lambda \notin \omega(p(T)) \Leftrightarrow p(T) - \lambda I = Weyl$$

$$\Leftrightarrow a_0(T - \mu_1 I) \dots . (T - \mu_1 I) = Weyl$$

$$\Leftrightarrow T - \mu, I = Weyl$$

for each $i = 1, 2, \dots, n$

$$\Leftrightarrow \mu, \notin \omega(T) \quad \text{for}$$

each $i = 1, 2, \dots, n$

$$\Leftrightarrow \lambda \notin p(\omega(T))$$

Which says that $p(\omega(T)) = p(\omega(T))$. If f is analytic on a neighbourhood of $\sigma(T)$, then there is a sequence (p_n) of polynomials such that $f_n \to f$ uniformly on $\sigma(T)$. Since $p_n(T)$ commutes with $f(T)$, by Oberai [8]

$$f(\omega(T)) = \lim_{p_n}(\omega(T)) = \lim \omega(p_n(T)) = \omega(f(T)).$$

Recall that $T \in B(H)$ is said to be *isoloid* if iso $\sigma(T) \subset \pi_0(T)$ (Oberai [9]).

Theorem 5

If T is dominant and f is analytic on a neighborhood of $\sigma(T)$, then

$$\omega(f(T)) = f(\omega(T))$$

Note one way spectral mapping theorem for analytic functions is f is analytic on neighbourhood then we state the following corollary;

Corollary 4

It was known that $\omega(T)$ satisfies the one-way spectral mapping theorem for analytic functions: if f is analytic on neighborhood of $\sigma(T)$ then;

$\omega(f(T) \subset f(\omega(T))$________________(1)

*B*u*t* specifically when we focus on a fredholm operator of index 0, (1) becomes;

$\omega(f(T) \subseteq f(\omega(T))$________________(2)

Taking continuity of $(\omega(T)$ alone and f(t) alone then the inequality

$(f(\omega(T)) \subseteq) \; \omega(\, f(T)$________________(3)

Lemma 1

Let $T \in B(H)$ be isoloid. Then for any polynomial

$p(t), p\big(\sigma(T) - \pi_{00}(T)\big) = \sigma\big(p(T)\big)$.

Let T be an $M -$ hyponormal operator which satisfies the additional property that for all z in the complex plane, all integers n and all x in H,

$\|(T-z)^n x\|^2 < M\|(T-z)^{2n}x\|\|x\|$

T is said to be an operator of M *−power class*(N). The following M −hyponormal operator T which is not hyponormal is of $M -$ power class (N) (Instratescu[7]):

Let $\{e_1\}$ be an orthonormal basis for H, and define

$$Te_1 \begin{cases} e_2, & \text{if } i = 1 \\ 2e_3, & \text{if } i = 2 \\ e_{,+1}, & \text{if } i \geq 3 \end{cases}$$

i.e., T is a weighted shift. From the definition of T we see that T is similar to the unilateral shift. Thus there exists an S such that $T = SU\,S^{-1}$. In our case $\|S\| = 2, \|S^{-1}\| = 1$. Since U is the unilateral shift, U is a hyponorma operator, and thus for every n and $z \in \mathbb{C}$ the operator $(U-z)^n$ is of class (N). It follows that $\|(U-z)^n x\|^2 \leq \|(U-z)^{2n}x\|$

For all $x \in H$ with $\|x\| = 1$, and hence T is of M −power class with $M = 4$. Thus our class is strickly larger than the class of hyponormal operators. Since $\omega(T) = \omega(U) = D$ (the closed unit disc) and $\pi_0(T) = \emptyset, \sigma(T) = \omega(T)$ and so Wely's theorem holds for T.

Theorem 6

If $T \in B(H)$ is an operator of $M -$ power class (N), the for any polynomial p on a neighbourhood of $\sigma(T)$ Weyl's theorem holds for $p(T)$.

Proof

T is isoloid and Weyl's theorem holds for any operator of $M -$ power class (N). Hence by Theorem 2 and Lemma 1,

$\omega\big(p(T)\big) = p\big(\omega(T)\big) = p(\sigma(T) - \pi_{00}(T)) - -\pi_{00}\big(p(T)\big)$

Therefore Weyl's theorem holds for $p(T)$.

Since every hypo normal operator is of 1 – power class (N), we obtain the following result which answers the question of Oberai.

Corollary 5

If $T \in B(\mathcal{H})$ is hyponormal, then for any polynomial p on a neighborhood of $\sigma(T)$ Weyl's theorem holds for $p(T)$.

CONCLUSION AND RECOMMENDATIONS

Conclusion: The following are the main results:

A power of a 2-isometry is again a 2-isometry.

It's now clear that a unilateral weighed shift is an isometry if all its weights lies on the unit circle.

It's also clear that if $T \in B(H)$ be isoloid. Then for any polynomial

$(t), p(\sigma(T) - \pi_{00}(T)) = \sigma(p(T))$.

T is said to be an operator of M *−power class*(N). The following M −hyponormal operator T which is not hyponormal is of M − power class (N) (Instratescu[7]):

Let $\{e_1\}$ be an orthonormal basis for H, and define

$$Te_1 \begin{cases} e_2, & \text{if } i = 1 \\ 2e_3, & \text{if } i = 2 \\ e_{,+1}, & \text{if } i \geq 3 \end{cases}$$

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$T = SU\,S^{-1}$. In our case $\|S\| = 2, \|S^{-1}\| = 1$. Since U is the unilateral shift, U is a hyponorma operator, and thus for every n and $z \in \mathbb{C}$ the operator $(U - z)^n$ is of class (N). It follows that $\|(U-z)^n x\|^2 \leq \|(U-z)^{2n} x\|$

For all $x \in H$ with $\|x\| = 1$, and hence T is of M −power class with $M = 4$. Thus our class is strickly larger than the class of hyponormal operators. Since $\omega(T) = \omega(U) = D$ (the closed unit disc) and $\pi_0(T) = \emptyset, \sigma(T) = \omega(T)$ and so Wely's theorem holds for T.

We also note that if T is dominant and f is analytic on a neighborhood of $\sigma(T)$, then

$$\omega(f(T)) = f(\omega(T)).$$

Also If S and $T$are dominant operators, then

S, T Weyl

$\Leftrightarrow ST$ Weyl.

Recommendations: In future, the study should focus on relationship of Weyl spectrum with respect to other components of the spectrum. Study on perfect inclusions of components of spectrum the study of the *essential spectrum* will take this study to another higher level.

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