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Spectrum of bounded operators in Hilbert spaces

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Abstract
This paper makes an attempt to study the spectrum operators by emphasising on condition of commuting operators so as to expose more properties in the classes of operators. Here, study of various classes of bounded operators on a Hilbert space H is one of the most important topics in the preparation of the study of the Hilbert spaces. In case a abounded operator A commutes at least with its own adjoint A* it forms important classes of operators on H, eg normal, unitary, self–adjoint etc. The operators under the study are bounded operators operating in a complete space called Hilbert spaces.

Key words: Bounded operators, commuting operators, Hilbert spaces, Spectrum, unitary.

INTRODUCTION

In the study, the spectrum in various classes of bounded operators on a Hilbert space H is one of the most important topics in the preparation of the study of the Hilbert spaces. One of the striking features of the collection of bounded operators on H is that very few of them commute with each other; i.e AB does not generally equal to BA for A, B $\in$ B(H). In case a abounded operator A commutes at least with its own adjoint A* and it forms important classes of operators on H, eg normal, unitary, self adjoint etc.

Definition 1

Let A $\in$ B(H)
A is called normal if $A^*A = AA^*$
unitary if $A^*A = I = AA^*$ ie $A^* = A^{-1}$
Self adjoint if $A^* = A$
Hyponomal if $A^*A \geq AA^*$

Remark 1

We first give various results on normal operators in relation to the spectrum.

Lamma 1

Let A$\in$B(H) be a normal operator. Then
a) If $\lambda \neq \mu$ ker $(\lambda I\text{-}A) \perp$ ker (UI-T)
b) For every $\lambda \in \not\subset$ ker $(\lambda I\text{-}A)$ and ker $((\lambda I\text{-}A)^{\perp}$ are invariant under both.
A and A*(Douglas, 1969).

Proof

a) If $Ax = \lambda x$ and $Ay = \mu y$ then
$A^*y = \mu y$ because ker A = ker A*
Hence we have;
$y \in \ker(\mu I\text{-}A) = \ker(\mu I\text{-}A^*)$

Therefore,
$\lambda <x,y> = <Ax,y> = <x, T^*y> = <x, \mu y> = \mu <x,y>$
$\lambda \neq \mu$
and so $<x,y>=0$
b) as $\lambda I\text{-}A$ commutes with
A and A*
Ker $(\lambda I\text{-}A)$ is invariant under both A and A* for all
$y \in \ker(\lambda I\text{-}A)$ we have
$<Ax,y> = <x, A^*y> =0$

Hence, $Ax \in (\ker(\lambda I\text{-}A)^{\perp}$
Similarly
$<A^*x,y> = <x, T_y> = 0$
for every $y \in \ker(\lambda I\text{-}T)$ and so
$T^*x = \ker(\lambda I\text{-}T)^{\perp}$

Theorem 1

Let A $\in$ B(H) be normal
a) if λ is an eigenvalue of A and x is corresponding eigenvector, then λ is an eigenvalue of A* and the same x is an eigenvector of A* corresponding to λ.
b) Eigenvectors of A corresponding to distinct eigenvalues are orthogonal.
c) Every spectral value of A is an approximate eigenvalue of A (Boullobas, 1990).

Proof

a) Let $Ax = \lambda x$ for some $\lambda \in \not\subset$ and $0 \neq x \in H$. Then by some results ie A is normal iff
$||A(x)|| = ||A^*x||$ for all $x \in H$
then;
$||A^*x - \lambda x|| = ||(A-\lambda I)^*x||$
$= ||(A - \lambda I)x|| = 0$

Hence, $A^*x = \lambda x$
This proves (a)

b) Let $Ax_1 = \lambda_1 x_1$ and
$Ax_2 = \lambda_2 x_2$ for some $\lambda_1 \neq \lambda_2$
In $\not\subset$ and $x_1, x_2 \in H$
Then by (a) above
$\_A^*x_2 = \lambda x_2$ so that

$\lambda_1 <x_1, x_2> = <\lambda_1 x_1, x_2>$
$= <Ax_1, x_2>$
$= <x_1, A^* x_2>$
$= <x_1, \lambda_2 x_2>$
$= \lambda_2 <x_1, x_2>$
Since $\lambda_1 \neq \lambda_2$ we see that
$<x_1, x_2> = 0$
This proves (b)

c) Let $\lambda \in \sigma(A)$
Then we know that
$\sigma(A) = \{\lambda : \lambda \in \sigma_p(A)\} \cup \{\lambda : \lambda \in \sigma_p(A^*)\}$
$= \{\lambda : \lambda \in \sigma_{ap}(A)\} \cup \{\lambda : \lambda \in \sigma_p(A^*)\}$
then either $\lambda \in \sigma_p(A^*)$ or $\lambda \in \sigma_{ap}(A)$
if $\lambda \in \sigma_p(A^*)$ then by (a)
above, $\lambda \in \sigma_{p(A)} \subset \sigma_{ap}(A)$

Thus, in any case, λ is
An approximate eigenvalue of A.
Thus in any case, λ is an approximate eigenvalues of A. **Q.E.D.**

Remark 2
We had considered $A \in (B)H$ being normal and so the prove of (c) above clearly shows that in any case, λ is an approximate eigenvalue of A, which is a very important result and so we state the following theorem.

Theorem 2
If $A \in B(H)$ is normal then $R\sigma(A) = \varnothing$ (Boullobas, 1990).

Proof
Now, $\lambda \in \mathbb{C}$, belongs to $R\sigma(A)$ if $(\lambda 1 - A)^{-1}$ exist as a map but $R(\lambda 1 - A) \neq H$
Let $\lambda \in \mathbb{C}$, be such that;
$R(\lambda I - A) \neq H$
We show that this condition implies that;
$\lambda \in \sigma_p(A)$ when A is normal
Which in turn implies; $R\sigma(A) = \varnothing$

Since $R(\lambda I - A) \neq H$ it follows that $R(\lambda I - A)^{\perp} \neq \{0\}$
$\Rightarrow N(\lambda I - A^*) \neq 0$ where N is the Null set. By use of $R(A)^{\perp} = N(A^*)$ i.e. $\exists x \in H$
such that $(\lambda I - A^*)x = 0$-----------------(*)
Since A normal, so is $\lambda I - A$.
This is $\equiv$ to the condition that;
$||(\lambda I - A)y|| = ||(\lambda I - A^*)y||$ for all $y \in H$------------------(**)
By (*) and (**) we have that $||(\lambda I - A)x|| = 0$ for some $x \neq 0 \rightarrow (\lambda I - A)x = 0$ for some $x \neq 0$.
i.e. $Ax = \lambda x$ has non-trivial roots in x.
i.e. $\lambda \in \sigma_p(A)$
Hence, $R\sigma(A) = \varnothing$ **Q.E.D**

Remark 3
Let us now turn to exhibit some results on self-adjoint operators and so we state the following theorem; which is very important.

Theorem 3
Let $A \in B(H)$ be self – adjoint. Then $\sigma(A) \subseteq R$. (Rimaye, 1981).

Proof
Let $\lambda \in \mathbb{C}$, with im $\lambda \neq 0$. Then for any $x \in H$ such that $x \neq 0$, we have that;
$0 < |\bar{\lambda} - \lambda| \, ||x||^2$
$= |<(A - \lambda I)x, x> - <(A - \bar{\lambda} I)x, x>|$
$= |<A - \lambda I) x, x> - <x, (A - \lambda I)x>|$
$\leq 2||(A - \lambda I)x|| \, ||x||$
if $\lambda \in \sigma(A) = \sigma_{ap}(A)$ then there exists a sequence of vector (x_n)
with $||x_n|| = 1$ such that
$(A - \lambda I)x_n \rightarrow 0$ — (a)
since $2||(A - \lambda I)x_n||$ must be greater than or equal to
$|A - \lambda|$ we have that
$\lim_{n \rightarrow \infty} ||(A - \lambda I)x_n|| \geq |\bar{\lambda} - \lambda| > 0$ — (b)
if im $\lambda \neq 0$

Thus, (a) and (b) are compatible
If λ is such that
$\lambda = \lambda$
im λ=0

Therefore, for λ to belong to
$\sigma_{ap}(A)$ we have that
im λ=0
$\therefore \sigma_{ap}(A) \subseteq R$______________ *
Or $\sigma_{ap}(A) \subseteq R$

Remark 4
We now exhibit results of unitary operators.

Theorem 4
A is unitary iff $||A(x)|| = ||x||$ for all $x \in H$ and A is onto (Rimaye, 1981).

Proof
If A is unitary, then for $x \in H$;
$\langle Ax, Ax\rangle = \langle A^*Ax, x\rangle = \langle x, x\rangle = ||x||^2$
Also, since A is invertible, the range of A is the whole of H. Conversely, assume that $||Ax|| = ||x||$ for all $x \in H$ and A is onto for $x \in H$
$\langle (A^*A-I)x, x)\rangle = \langle Ax, Ax\rangle - \langle x, x\rangle$
$= ||Ax||^2 - ||x||^2 = 0$
Since $(A^*A-I) = A^*A-I$. To prove $AA^*=I$ we show that A is invertible in B(H) if A(x) = 0 then $||x|| = ||A(x)|| = 0$, so that A is onto, $A^{-1}:H \to H$;
Is a well defined map and it is linear.
Also for $y \in H$, if A(x) = y
then $||A^{-1}(y)|| = ||A^{-1}A(x)|| = ||y||$
Hence, A^{-1} is bounded and belongs to B(H) ie A is invertible in B(H).
Now $AA^* = (AA^*)(AA^{-1}) = A(A^*A)^{-1} = I$
and hence A is unitary. **Q.E.D.**

Theorem 5
The spectrum of unitary operator U lies entirely on the unit circle (Boullobas, 1990).

Proof
i) Since $||U||=1$, it follows that if $|\lambda|>||A||$, the operator λI-A is invertible, then the spectrum of U is confined to the closed unit disc, $\{\lambda | |\lambda| \leq 1\}$.
ii) Let $|\mu|<|$. Then for any non-zero vector g, we have
$||\mu g||=||g||>||\mu g||$, and so
$(\mu I-U)g \neq 0$
Thus, μI - U is one– to-one.
iii) Assuming as in (ii), that $|\mu|<1$, we shall show that the range of $(\mu I-U)$ is dense in H. If this were not so, there would exist a non-zero vector h such that $\langle(\mu I-U)g, h\rangle=0$ for every vector g. Choosing g the vector $\cup^*h$, we would obtain,
$\langle\mu\cup^*h, h\rangle = \langle\cup\cup^*h, h\rangle = \langle h, h\rangle$
and by the Schwartz inequality we obtain $||h||^2 \leq |\mu| . || U^*|| . ||h||^2$. Dividing by $||h||^2$ and recalling that $||U^*|| =1$,
We would obtain $|\mu| \geq 1$, contradicting our hypothesis.
iv) Continuing to assume that $|\mu|<1$, we can extend the result (of) (iii) to show that the range of μI-U consists of all of H.
Hence, μI-U is invertible, and so the spectrum is confined entirely to the circumference of the unit disc. **Q.E.D.**

RESULTS AND CONCLUSION
The spectrum of unitary operator U lies entirely on the unit circle.
A is unitary iff $||A(x)|| = ||x||$ for all $x \in H$ and A is onto mapping.
Since $||U||=1$, it follows that if $|\lambda|>||A||$, the operator λI-A is invertible, then the spectrum of U is confined to the closed unit disc, $\{\lambda : |\lambda| \leq 1\}$.
Also If $A \in B(H)$ is normal then $R\sigma(A)=\emptyset$.

REFERENCES

1. Bollobas, B. (1990). Linear Analysis: An Introductory Course. Cambridge University Press.
2. Douglas, R. G. (1969). On the operator equation $S^*XT = X$ and related topics, *Acta Sci. Maths (Szeged), 30*, 19-32.
3. Limaye, B. V. (1981). Functional Analysis. J. Wiley.